Ricardo López-Ruiz
Héctor Mancini
Xavier Calbet

A Statistical Measure of Complexity

– Book Chapter –

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Chapter 1
A Statistical Measure of Complexity

Abstract *In this chapter, a statistical measure of complexity is introduced and some of its properties are discussed. Also, some straightforward applications are shown.*

1.1 Shannon Information

Entropy plays a crucial theoretical role in physics of macroscopic equilibrium systems. The probability distribution of accessible states of a constrained system in equilibrium can be found by the inference principle of maximum entropy [1]. The macroscopic magnitudes and the laws that relate them can be calculated with this probability distribution by standard statistical mechanics techniques.

The same scheme could be thought for extended systems far from equilibrium, but in this case we do have neither a method to find the probability distribution nor the knowledge of the relevant magnitudes bringing the information that can predict the system's behavior. It is not the case, for instance, with the metric properties of low dimensional chaotic systems by means of the Lyapunov exponents, invariant measures and fractal dimensions [2].

Shannon information or entropy H [3] can still be used as a magnitude in a general situation with N accessible states:

$$H = -K \sum_{i=1}^{N} p_i \log p_i \qquad (1.1)$$

with K a positive real constant and p_i the normalized associated probabilities, $\sum_{i=1}^{N} p_i = 1$. An isolated system in equilibrium presents equiprobability, $p_i = 1/N$ for all i, among its accessible states and this is the situation of maximal entropy,

$$H_{max} = K \log N. \qquad (1.2)$$

If the system is out of equilibrium, the entropy H can be expanded around this maximum H_{max}:

$$H(p_1, p_2, \ldots, p_N) = K \log N - \frac{NK}{2} \sum_{i=1}^{N} \left(p_i - \frac{1}{N} \right)^2 + \ldots = H_{max} - \frac{NK}{2} D + \ldots \qquad (1.3)$$

where the quantity $D = \sum_i (p_i - 1/N)^2$, that we call *disequilibrium*, is a kind of distance from the actual system configuration to the equilibrium. If the expression (1.3) is multiplied by H we obtain:

$$H^2 = H \cdot H_{max} - \frac{NK}{2} H \cdot D + K^2 f(N, p_i), \qquad (1.4)$$

where $f(N, p_i)$ is the entropy multiplied by the rest of the Taylor expansion terms, which present the form $\frac{1}{N} \sum_i (N p_i - 1)^m$ with $m > 2$. If we rename $C = H \cdot D$,

$$C = cte \cdot H \cdot (H_{max} - H) + K \bar{f}(N, p_i), \qquad (1.5)$$

with $cte^{-1} = NK/2$ and $\bar{f} = 2f/N$. The idea of distance for the disequilibrium is now clearer if we see that D is just the real distance $D \sim (H_{max} - H)$ for systems in the vicinity of the equiprobability. In an ideal gas we have $H \sim H_{max}$ and $D \sim 0$,

then $C \sim 0$. Contrarily, in a crystal $H \sim 0$ and $D \sim 1$, but also $C \sim 0$. These two systems are considered as classical examples of simple models and are extrema in a scale of disorder (H) or disequilibrium (D) but those should present null complexity in a hypothetic measure of *complexity*. This last asymptotic behavior is verified by the variable C (Fig. 1.1) and C has been proposed as a such magnitude [4]. We formalize this simple idea recalling the recent definition of *LMC complexity* in the next section.

Let us see another important property [5] arising from relation (1.5). If we take the time derivative of C in a neighborhood of equilibrium by approaching $C \sim H(H_{max} - H)$, then we have

$$\frac{dC}{dt} \sim -H_{max}\frac{dH}{dt}. \qquad (1.6)$$

The irreversibility property of H implies that $\frac{dH}{dt} \geq 0$, the equality occurring only for the equipartition, therefore

$$\frac{dC}{dt} \leq 0. \qquad (1.7)$$

Hence, in the vicinity of H_{max}, LMC complexity is always decreasing on the evolution path towards equilibrium, independently of the kind of transition and of the system under study. This does not forbid that complexity can increase when the system is very far from equilibrium. In fact this is the case in a general situation as it can be seen, for instance, in the gas system presented in Ref. [6].

1.2 A Statistical Complexity Measure

On the most basic grounds, an object, a procedure, or system is said to be "complex" when it does not match patterns regarded as simple. This sounds rather like an oxymoron but common knowledge tells us what is simple and complex: simplified systems or idealizations are always a starting point to solve scientific problems. The notion of "complexity" in physics [7, 8] starts by considering the perfect crystal and the isolated ideal gas as examples of simple models and therefore as systems with zero "complexity". Let us briefly recall their main characteristics with "order", "information" and "equilibrium".

A perfect crystal is completely ordered and the atoms are arranged following stringent rules of symmetry. The probability distribution for the states accessible to the perfect crystal is centered around a prevailing state of perfect symmetry. A small piece of "information" is enough to describe the perfect crystal: the distances and the symmetries that define the elementary cell. The "information" stored in this system can be considered minimal. On the other hand, the isolated ideal gas is completely disordered. The system can be found in any of its accessible states with the same probability. All of them contribute in equal measure to the "information" stored in the ideal gas. It has therefore a maximum "information". These two simple systems

are extrema in the scale of "order" and "information". It follows that the definition of "complexity" must not be made in terms of just "order" or "information".

It might seem reasonable to propose a measure of "complexity" by adopting some kind of distance from the equiprobable distribution of the accessible states of the system [4]. Defined in this way, "disequilibrium" would give an idea of the probabilistic hierarchy of the system. "Disequilibrium" would be different from zero if there are privileged, or more probable, states among those accessible. But this would not work. Going back to the two examples we began with, it is readily seen that a perfect crystal is far from an equidistribution among the accessible states because one of them is totally prevailing, and so "disequilibrium" would be maximum. For the ideal gas, "disequilibrium" would be zero by construction. Therefore such a distance or "disequilibrium" (a measure of a probabilistic hierarchy) cannot be directly associated with "complexity".

In Figure 1.1 we sketch an intuitive qualitative behavior for "information" H and "disequilibrium" D for systems ranging from the perfect crystal to the ideal gas. As indicated in the former section, this graph suggests that the product of these two quantities could be used as a measure of "complexity": $C = H \cdot D$. The function C has indeed the features and asymptotic properties that one would expect intuitively: it vanishes for the perfect crystal and for the isolated ideal gas, and it is different from zero for the rest of the systems of particles. We will follow these guidelines to establish a quantitative measure of "complexity".

Before attempting any further progress, however, we must recall that "complexity" cannot be measured univocally, because it depends on the nature of the description (which always involves a reductionist process) and on the scale of observation. Let us take an example to illustrate this point. A computer chip can look very different at different scales. It is an entangled array of electronic elements at microscopic scale but only an ordered set of pins attached to a black box at a macroscopic scale.

We shall now discuss a measure of "complexity" based on the statistical description of systems. Let us assume that the system has N accessible states $\{x_1, x_2, ..., x_N\}$ when observed at a given scale. We will call this an N-system. Our understanding of the behavior of this system determines the corresponding probabilities $\{p_1, p_2, ..., p_N\}$ (with the condition $\sum_{i=1}^{N} p_i = 1$) of each state ($p_i > 0$ for all i). Then the knowledge of the underlying physical laws at this scale is incorporated into a probability distribution for the accessible states. It is possible to find a quantity measuring the amount of "information". As presented in the former section, under to the most elementary conditions of consistency, Shannon [3] determined the unique function $H(p_1, p_2, ..., p_N)$ given by expression (1.1), that accounts for the "information" stored in a system, where K is a positive constant. The quantity H is called *information*. The redefinition of information H as some type of monotone function of the Shannon entropy can be also useful in many contexts. In the case of a crystal, a state x_c would be the most probable $p_c \sim 1$, and all others x_i would be very improbable, $p_i \sim 0$ $i \neq c$. Then $H_c \sim 0$. On the other side, equiprobability characterizes an isolated ideal gas, $p_i \sim 1/N$ so $H_g \sim K \log N$, i.e., the maximum of information for a N-system. (Notice that if one assumes equiprobability and $K = \kappa \equiv Boltzmann$
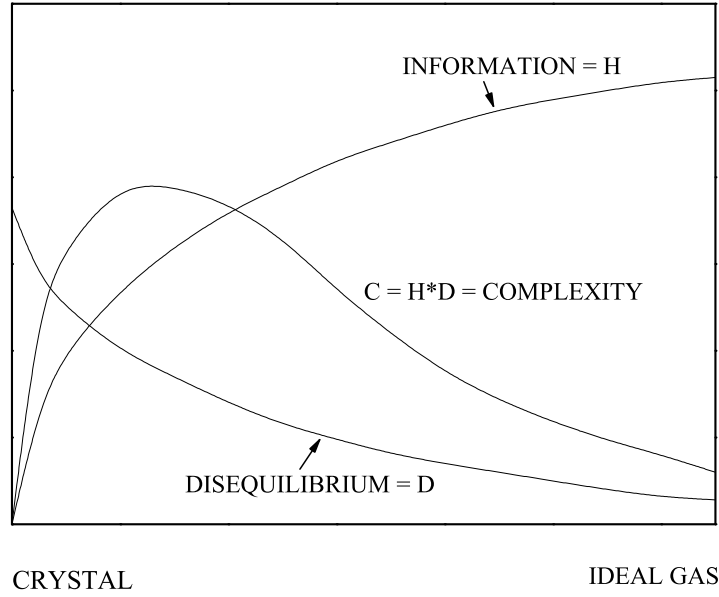
Fig. 1.1 Sketch of the intuitive notion of the magnitudes of "information" (H) and "disequilibrium" (D) for the physical systems and the behavior intuitively required for the magnitude "complexity". The quantity $C = H \cdot D$ is proposed to measure such a magnitude.

constant, H is identified with the thermodinamic entropy, $S = \kappa \log N$). Any other N-system will have an amount of information between those two extrema.

Let us propose a definition of *disequilibrium D* in a *N*-system [9]. The intuitive notion suggests that some kind of distance from an equiprobable distribution should be adopted. Two requirements are imposed on the magnitude of D: $D > 0$ in order to have a positive measure of "complexity" and $D = 0$ on the limit of equiprobability. The straightforward solution is to add the quadratic distances of each state to the equiprobability as follows:

$$D = \sum_{i=1}^{N} \left(p_i - \frac{1}{N} \right)^2 . \tag{1.8}$$

According to this definition, a crystal has maximum disequilibrium (for the dominant state, $p_c \sim 1$, and $D_c \to 1$ for $N \to \infty$) while the disequilibrium for an ideal gas vanishes ($D_g \sim 0$) by construction. For any other system D will have a value between these two extrema.

We now introduce the definition of *complexity* C of a N-system [4, 10]. This is simply the interplay between the information stored in the system and its disequilibrium:

$$C = H \cdot D = -\left(K \sum_{i=1}^{N} p_i \log p_i \right) \cdot \left(\sum_{i=1}^{N} \left(p_i - \frac{1}{N} \right)^2 \right). \qquad (1.9)$$

This definition fits the intuitive arguments. For a crystal, disequilibrium is large but the information stored is vanishingly small, so $C \sim 0$. On the other hand, H is large for an ideal gas, but D is small, so $C \sim 0$ as well. Any other system will have an intermediate behavior and therefore $C > 0$.

As was intuitively suggested, the definition of complexity (1.9) also depends on the *scale*. At each scale of observation a new set of accessible states appears with its corresponding probability distribution so that complexity changes. Physical laws at each level of observation allow us to infer the probability distribution of the new set of accessible states, and therefore different values for H, D and C will be obtained. The straightforward passage to the case of a continuum number of states, x, can be easily inferred. Thus we must treat with probability distributions with a continuum support, $p(x)$, and normalization condition $\int_{-\infty}^{+\infty} p(x)dx = 1$. Disequilibrium has the limit $D = \int_{-\infty}^{+\infty} p^2(x)dx$ and the complexity could be defined by:

$$C = H \cdot D = -\left(K \int_{-\infty}^{+\infty} p(x) \log p(x)dx \right) \cdot \left(\int_{-\infty}^{+\infty} p^2(x)dx \right). \qquad (1.10)$$

Other possibilities for the continuous extension of C are also possible. For instance, a successful attempt of extending the LMC complexity for continuous systems has been performed in Ref. [11]. When the number of states available for a system is a continuum then the natural representation is a continuous distribution. In this case, the entropy can become negative. The positivity of C for every distribution is recovered by taking the exponential of H [12]. If we define $\hat{C} = \hat{H} \cdot D = e^H \cdot D$ as an extension of C to the continuous case interesting properties characterizing the indicator $\hat{C}$ appear. Namely, its invariance under translations, rescaling transformations and replication convert $\hat{C}$ in a good candidate to be considered as an indicator bringing essential information about the statistical properties of a continuous system.

Direct simulations of the definition give the values of C for general N-systems. The set of all the possible distributions $\{p_1, p_2, ..., p_N\}$ where an N-system could be found is sampled. For the sake of simplicity H is normalized to the interval $[0,1]$. Thus $H = \sum_{i=1}^{N} p_i \log p_i / \log N$. For each distribution $\{p_i\}$ the normalized information $H(\{p_i\})$, and the disequilibrium $D(\{p_i\})$ (eq. 1.8) are calculated. In each case the normalized complexity $C = H \cdot D$ is obtained and the pair (H,C) stored. These two magnitudes are plotted on a diagram $(H,C(H))$ in order to verify the qualitative behavior predicted in Figure 1.1. For $N = 2$ an analytical expression for the curve $C(H)$ is obtained. If the probability of one state is $p_1 = x$, that of the second one is simply $p_2 = 1 - x$. Complexity vanishes for the two simplest 2-systems: the crystal ($H = 0$; $p_1 = 1$, $p_2 = 0$) and the ideal gas ($H = 1$; $p_1 = 1/2$, $p_2 = 1/2$). Let us notice that this curve is the simplest one that fulfills all the conditions discussed in

N = 3
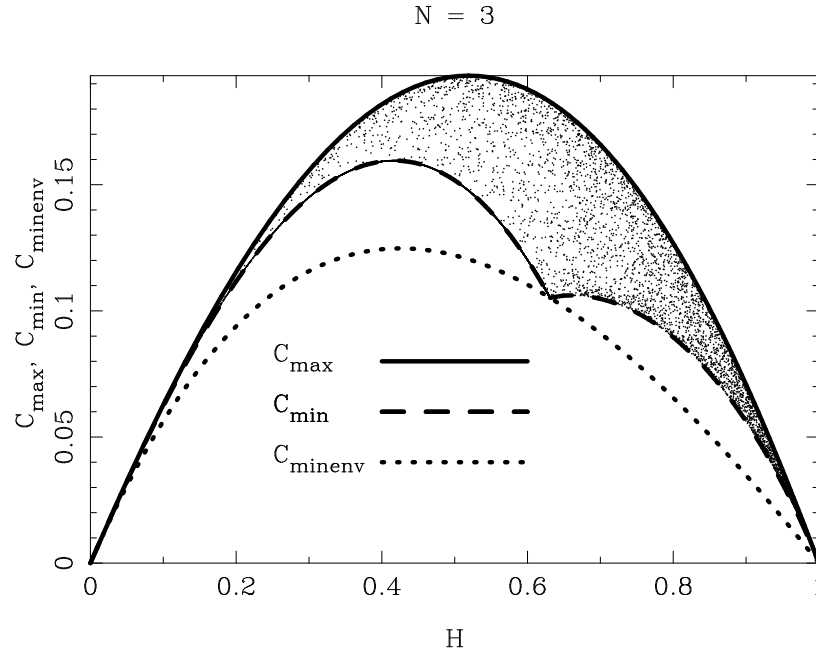
Fig. 1.2 In general, dependence of complexity (C) on normalized information (H) is not univocal: many distributions $\{p_i\}$ can present the same value of H but different C. This is shown in the case $N = 3$.

the introduction. The largest complexity is reached for $H \sim 1/2$ and its value is: $C(x \sim 0.11) \sim 0.151$. For $N > 2$ the relationship between H and C is not univocal anymore. Many different distributions $\{p_i\}$ store the same information H but have different complexity C. Figure 1.2 displays such a behavior for $N = 3$. If we take the maximum complexity $C_{max}(H)$ associated with each H a curve similar to the one for a 2-system is recovered. Every 3-system will have a complexity below this line and upper the line of $C_{min}(H)$ and also upper the minimum envelope complexity C_{minenv}. These lines will be analytically found in a next section. In Figure 1.3 curves $C_{max}(H)$ for the cases $N = 3,\ldots,10$ are also shown. Let us observe the shift of the complexity-curve peak to smaller values of entropy for rising N. This fact agrees with the intuition telling us that the biggest complexity (number of possibilities of 'complexification') be reached for lesser entropies for the systems with bigger number of states.

Let us return to the point at which we started this discussion. Any notion of complexity in physics [7, 8] should only be made on the basis of a well defined or operational magnitude [4, 10]. But two additional requirements are needed in order to obtain a good definition of complexity in physics: (1) the new magnitude must be measurable in many different physical systems and (2) a comparative relationship and a physical interpretation between any two measurements should be possible.
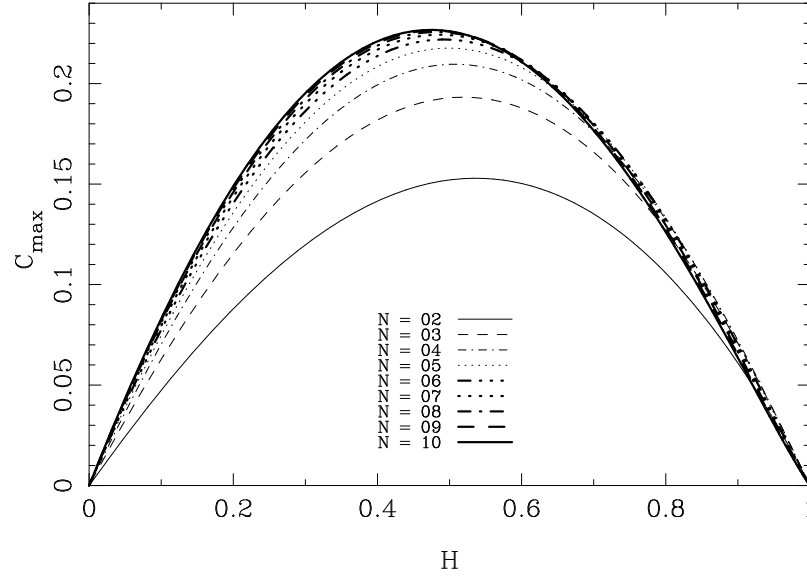
Fig. 1.3 Complexity ($C = H \cdot D$) as a function of the normalized information (H) for a system with two accessible states ($N = 2$). Also curves of maximum complexity (C_{max}) are shown for the cases: $N = 3, \ldots, 10$.

Many different definitions of complexity have been proposed to date, mainly in the realm of physical and computational sciences. Among these, several can be cited: algorithmic complexity (Kolmogorov-Chaitin) [13, 14], the Lempel-Ziv complexity [15], the logical depth of Bennett [16], the effective measure complexity of Grassberger [17], the complexity of a system based in its diversity [18], the thermodynamical depth [19], the ε-machine complexity [20] , the physical complexity of genomes [21], complexities of formal grammars, etc. The definition of complexity (1.9) proposed in this section offers a new point of view, based on a statistical description of systems at a given *scale*. In this scheme, the knowledge of the physical laws governing the dynamic evolution in that scale is used to find its accessible states and its probability distribution. This process would immediately indicate the value of complexity. In essence this is nothing but an interplay between the information stored by the system and the *distance from equipartition* (measure of a probabilistic hierarchy between the observed parts) of the probability distribution of its accessible states. Besides giving the main features of a "intuitive" notion of complexity, we will show in this chapter that we can go one step further and that it is possible to compute this quantity in relevant physical situations [6, 22, 23]. The most important point is that the new definition successfully enables us to discern situations regarded as complex.

1.3 LMC Complexity: Extremal Distributions

Now we proceed to calculate the distributions which maximize and minimize the LMC complexity and its asymptotic behavior [6].

Let us assume that the system can be in one of its N possible accessible states, i. The probability of the system being in state i will be given by the discrete distribution function, $f_i \geq 0$, with the normalization condition $I \equiv \sum_{i=1}^{N} f_i = 1$. The system is defined such that, if isolated, it will reach equilibrium, with all the states having equal probability, $f_e = \frac{1}{N}$. Since we are supposing that H is normalized, $0 \leq H \leq 1$, and $0 \leq D \leq (N-1)/N$, then complexity, C, is also normalized, $0 \leq C \leq 1$.

When an isolated system evolves with time, the complexity cannot have any possible value in a C versus H map as it can be seen in Fig. 1.2, but it must stay within certain bounds, $C_{\max}$ and $C_{\min}$. These are the maximum and minimum values of C for a given H. Since $C = D \cdot H$, finding the extrema of C for constant H is equivalent to finding the extrema of D.

There are two restrictions on D: the normalization, I, and the fixed value of the entropy, H. To find these extrema undetermined Lagrange multipliers are used. Differentiating expressions of D, I and H, we obtain

$$\frac{\partial D}{\partial f_j} = 2(f_j - f_e)\,, \tag{1.11}$$

$$\frac{\partial I}{\partial f_j} = 1\,, \tag{1.12}$$

$$\frac{\partial H}{\partial f_j} = -\frac{1}{\ln N}\left(\ln f_j + 1\right). \tag{1.13}$$

Defining λ_1 and λ_2 as the Lagrange multipliers, we get:

$$2(f_j - f_e) + \lambda_1 + \lambda_2(\ln f_j + 1)/\ln N = 0\,. \tag{1.14}$$

Two new parameters, α and β, which are a linear combinations of the Lagrange multipliers are defined:

$$f_j + \alpha \ln f_j + \beta = 0\,, \tag{1.15}$$

where the solutions of this equation, f_j, are the values that minimize or maximize the disequilibrium.

In the maximum complexity case there are two solutions, f_j, to Eq. (1.15) which are shown in Table 1.1. One of these solutions, $f_{\max}$, is given by

$$H = -\frac{1}{\ln N}\left[f_{\max} \ln f_{\max} + (1 - f_{\max})\ln\left(\frac{1 - f_{\max}}{N-1}\right)\right], \tag{1.16}$$

and the other solution by $(1 - f_{\max})/(N-1)$. The maximum disequilibrium, $D_{\max}$, for a fixed H is

$$D_{\max} = (f_{\max} - f_e)^2 + (N-1)\left(\frac{1-f_{\max}}{N-1} - f_e\right)^2 , \tag{1.17}$$

and thus, the maximum complexity, which depends only on H, is

$$C_{\max}(H) = D_{\max} \cdot H . \tag{1.18}$$

The behavior of the maximum value of complexity versus $\ln N$ was computed in Ref. [24].

Table 1.1 Probability values, f_j, that give a maximum of disequilibrium, $D_{\max}$, for a given H.

Number of states with f_j	f_j	Range of f_j
1	$f_{\max}$	$\frac{1}{N} \ldots 1$
$N-1$	$\frac{1-f_{\max}}{N-1}$	$0 \ldots \frac{1}{N}$

Table 1.2 Probability values, f_j, that give a minimum of disequilibrium, $D_{\min}$, for a given H.

Number of states with f_j	f_j	Range of f_j
n	0	0
1	$f_{\min}$	$0 \ldots \frac{1}{N-n}$
$N-n-1$	$\frac{1-f_{\min}}{N-n-1}$	$\frac{1}{N-n} \ldots \frac{1}{N-n-1}$

n can have the values $0, 1, \ldots N-2$.

Equivalently, the values, f_j, that give a minimum complexity are shown in Table 1.2. One of the solutions, $f_{\min}$, is given by

$$H = -\frac{1}{\ln N}\left[f_{\min}\ln f_{\min} + (1-f_{\min})\ln\left(\frac{1-f_{\min}}{N-n-1}\right)\right], \tag{1.19}$$

where n is the number of states with $f_j = 0$ and takes a value in the range $n = 0, 1, \ldots, N-2$. The resulting minimum disequilibrium, $D_{\min}$, for a given H is,

$$D_{\min} = (f_{\min} - f_e)^2 + (N-n-1)\left(\frac{1-f_{\min}}{N-n-1} - f_e\right)^2 + nf_e^2 . \tag{1.20}$$

Note that in this case $f_j = 0$ is an additional hidden solution that stems from the positive restriction in the f_i values. To obtain these solutions explicitly we can define x_i such that $f_i \equiv x_i^2$. These x_i values do not have the restriction of positivity imposed to f_i and can take a positive or negative value. If we repeat the Lagrange multiplier method with these new variables a new solution arises: $x_j = 0$, or equivalently, $f_j =$

0. The resulting minimum complexity, which again only depends on H, is

$$C_{min}(H) = D_{min} \cdot H. \tag{1.21}$$

As an example, the maximum and minimum of complexity, C_{max} and C_{min}, are plotted as a function of the entropy, H, in Fig. 1.4 for $N = 4$. Also, in this figure, it is shown the minimum envelope complexity, $C_{minenv} = D_{minenv} \cdot H$, where D_{minenv} is defined below. In Fig. 1.5 the maximum and minimum disequilibrium, D_{max} and D_{min}, versus H are also shown.
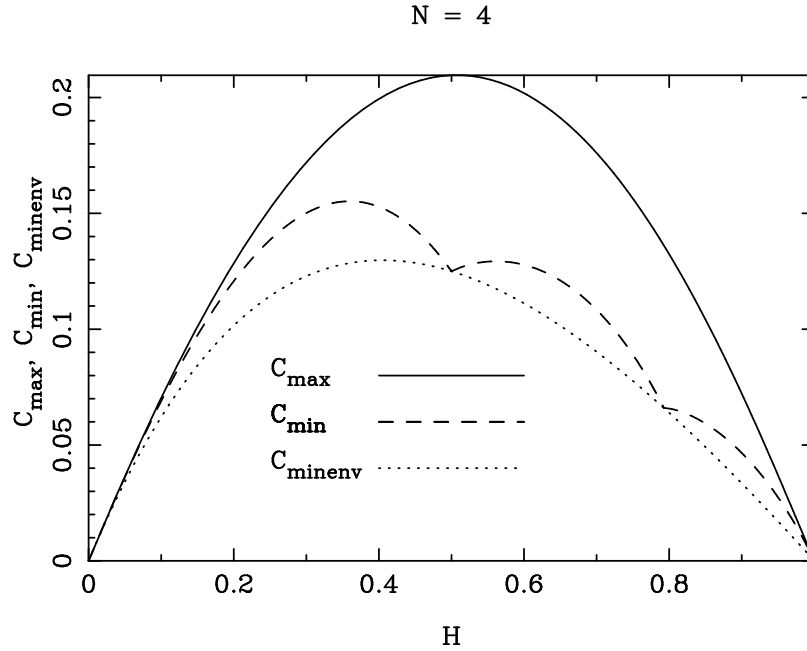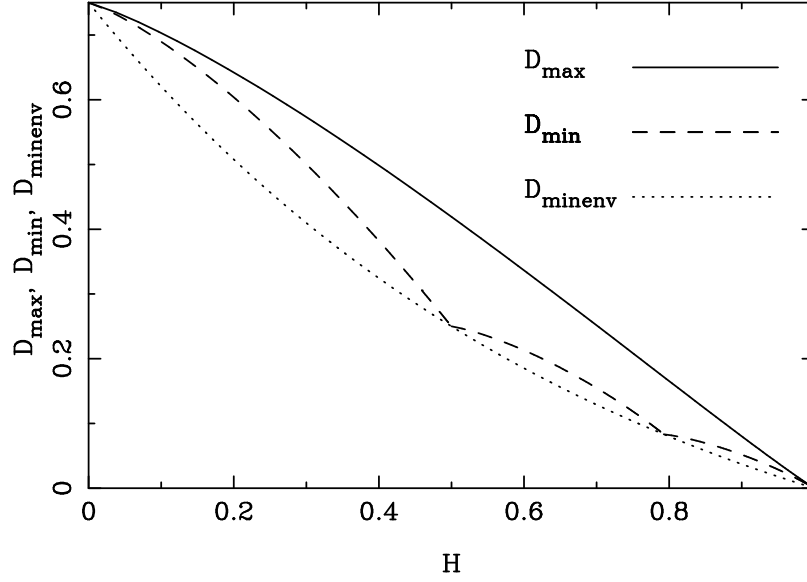

Fig. 1.4 Maximum, minimum, and minimum envelope complexity, C_{max}, C_{min}, and C_{minenv} respectively, as a function of the entropy, H, for a system with $N = 4$ accessible states.

As shown in Fig. 1.5 the minimum disequilibrium function is piecewise defined, having several points where its derivative is discontinuous. Each of these function pieces corresponds to a different value of n (Table 1.2).In some circumstances it might be helpful to work with the "envelope" of the minimum disequilibrium function. The function, D_{minenv}, that traverses all the discontinuous derivative points in the D_{min} versus H plot is

$$D_{minenv} = e^{-H \ln N} - \frac{1}{N}, \tag{1.22}$$

and is also shown in Figure 1.5.

N = 4

Fig. 1.5 Maximum, minimum, and minimum envelope disequilibrium, D_{max}, D_{min}, and D_{minenv} respectively, as a function of the entropy, H, for a system with $N = 4$ accessible states.

When N tends toward infinity the probability, f_{max}, of the dominant state has a linear dependence with the entropy,

$$\lim_{N \to \infty} f_{max} = 1 - H, \qquad (1.23)$$

and thus the maximum disequilibrium scales as $\lim_{N \to \infty} D_{max} = (1-H)^2$. The maximum complexity tends to

$$\lim_{N \to \infty} C_{max} = H \cdot (1-H)^2. \qquad (1.24)$$

The limit of the minimum disequilibrium and complexity vanishes, $\lim_{N \to \infty} D_{minenv} = 0$, and thus

$$\lim_{N \to \infty} C_{min} = 0. \qquad (1.25)$$

In general, in the limit $N \to \infty$, the complexity is not a trivial function of the entropy, in the sense that for a given H there exists a range of complexities between 0 and C_{max}, given by Eqs. (1.25) and (1.24), respectively.

In particular, in this asymptotic limit, the maximum of C_{max} is found when $H = 1/3$, or equivalently $f_{max} = 2/3$, which gives a maximum of the maximum complexity of $C_{max} = 4/27$. This value was numerically calculated in Ref. [24].

1.4 Rényi Entropies and LMC Complexity

Generalized entropies were introduced by Rényi [25] in the form of

$$I_q = \frac{1}{1-q} \log \left(\sum_{i=1}^{N} p_i^q \right), \tag{1.26}$$

where q is an index running over all the integer values. By differentiating I_q with respect to q a negative quantity is obtained independently of q, then I_q monotonously decreases when q increases.

The Rényi entropies are an extension of the Shannon information H. In fact, H is obtained in the limit $q \rightarrow 1$:

$$H = I_1 = \lim_{q \to 1} I_q = - \sum_{i=1}^{N} p_i \log p_i, \tag{1.27}$$

where the constant K of Eq. (1.1) is considered to be the unity. The disequilibrium D is also related with $I_2 = -\log \left(\sum_{i=1}^{N} p_i^2 \right)$. We have that

$$D = \sum_{i=1}^{N} p_i^2 - \frac{1}{N} = e^{-I_2} - \frac{1}{N}, \tag{1.28}$$

then the LMC complexity is

$$C = H \cdot D = I_1 \cdot \left(e^{-I_2} - \frac{1}{N} \right). \tag{1.29}$$

The behavior of C in the neighborhood of H_{max} takes the form

$$C \sim \frac{1}{N} (\log^2 N - I_1 I_2), \tag{1.30}$$

The obvious generalization of the Rényi entropies for a normalized continuous distribution $p(x)$ is

$$I_q = \frac{1}{1-q} \log \int [p(x)]^q dx. \tag{1.31}$$

Hence,

$$H = I_1 = - \int p(x) \log p(x) dx, \tag{1.32}$$

$$D = e^{-I_2} = \int [p(x)]^2 dx. \tag{1.33}$$

The dependence of $\hat{C} = e^H \cdot D$ with I_1 and I_2 yields

$$\log \hat{C} = (I_1 - I_2). \tag{1.34}$$

This indicates that a family of different indicators could derive from the differences established among Rényi entropies with different q-indices [5]. Let us remark at this point the coincidence of the indicator $\log \hat{C}$ with the quantity S_{str} introduced by Varga and Pipek as a meaningful parameter to characterize the shape of a distribution. They apply this formalism to the Husimi representation, i.e., to the projection of wave functions onto the coherent state basis [26]. A further generalization of the LMC complexity measure as function of the Rényi entropies has been introduced in Ref. [27].

The invariance of $\hat{C}$ under rescaling transformations implies that this magnitude is conserved in many different processes. For instance, the initial Gaussian-like distribution will continue to be Gaussian in a classical diffusion process. Then $\hat{C}$ is constant in time: $\frac{d\hat{C}}{dt} = 0$, and we have:

$$\frac{dI_1}{dt} = \frac{dI_2}{dt}.$$
(1.35)

The equal losing rate of I_1 and I_2, i.e., the synchronization of both quantities, is the cost to be paid in order to maintain the shape of the distribution associated to the system and, hence, all its statistical properties will remain unchanged during its time evolution.

1.5 Some Applications

If by complexity it is to be understood that property present in all systems attached under the epigraph of 'complex systems', this property should be reasonably quantified by the measures proposed in the different branches of knowledge. In our case, the main advantage of LMC complexity is its generality and the fact that it is operationally simple and do not require a big amount of calculations [28]. This advantage has been worked out in different examples, such as the study of the time evolution of C for a simplified model of an isolated gas, the "tetrahedral gas" [6] or also in the case of a more realistic gas of particles [29, 30], the slight modification of C as an effective method by which the complexity in hydrological systems can be identified [31], the attempt of generalize C in a family of simple complexity measures [32, 33, 34], some statistical features of the behavior of C for DNA sequences [35] or earthquake magnitude time series [36], some wavelet-based informational tools used to analyze the brain electrical activity in epilectic episodes in the plane of coordinates (H,C) [37], a method to discern complexity in two-dimensional patterns [38] or some calculations done on quantum systems [39, 40, 41, 42, 43]. As an example, we show in the next subsections some straightforward calculation of the LMC complexity [44].

1.5.1 Canonical ensemble

Each physical situation is closely related to a specific distribution of microscopic states. Thus, an isolated system presents equipartition, by hypothesis: the microstates compatible with a macroscopic situation are equiprobable [45]. The system is said to be in equilibrium. For a system surrounded by a heat reservoir the probability of the microstates associated to the thermal equilibrium follow the Boltzmann distribution. Let us try to analyze the behavior of C in an ideal gas in thermal equilibrium. In this case the probability p_i of each accesible state is given by the Boltzmann distribution:

$$p_i = \frac{e^{-\beta E_i}}{Q_N}, \tag{1.36}$$

$$Q_N = \int e^{-\beta E(p,q)} \frac{d^{3N} p \, d^{3N} q}{N! h^{3N}} = e^{-\beta A(V,T)}, \tag{1.37}$$

where Q_N is the partition function of the canonical ensemble, $\beta = 1/\kappa T$ with κ the Boltzmann constant and T the temperature, V the volume, N the number of particles, $E(p,q)$ the hamiltonian of the system, h is the Planck constant and $A(V,T)$ the Helmholtz potential.

Calculation of H and D gives us:

$$H(V,T) = (1 + T \frac{\partial}{\partial T})(\kappa \log Q_N) = S(V,T), \tag{1.38}$$

$$D(V,T) = e^{2\beta \, [A(V,T) - A(V,T/2)]}. \tag{1.39}$$

Note that Shannon information H coincides with the thermodynamic entropy S when K is identified with κ. If a system verifies the relation $U = C_v T$ (U the internal energy, C_v the specific heat) the complexity takes the form:

$$C(V,T) \sim cte(V) \cdot S(V,T) e^{-S(V,T)/\kappa} \tag{1.40}$$

that matches the intuitive function proposed in Figure 1.1.

1.5.2 Gaussian and exponential distributions

Gaussian distribution: Suppose a continuum of states represented by the x variable whose probability density $p(x)$ is given by the normal distribution of variance σ:

$$p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left(-\frac{x^2}{2\sigma^2} \right). \tag{1.41}$$

After calculating H and D, the expression for C is the following:

$$C_g = H \cdot D = \frac{K}{2\sigma\sqrt{\pi}}\left(\frac{1}{2} + \log(\sigma\sqrt{2\pi})\right). \tag{1.42}$$

If we impose the additional condition $H \geq 0$, then $\sigma \geq \sigma_{min} = (2\pi e)^{-1/2}$. The highest complexity is reached for a determined width: $\bar{\sigma} = \sqrt{(e/2\pi)}$.

Exponencial distribution: Consider an exponencial distribution of variance γ:

$$p(x) = \begin{cases} \frac{1}{\gamma}e^{-x/\gamma} & x > 0, \\ 0 & x < 0. \end{cases} \tag{1.43}$$

The same calculation gives us:

$$C_e = \frac{K}{2\gamma}(1 + \log\gamma), \tag{1.44}$$

with the condition $H \geq 0$ imposing $\gamma \geq \gamma_{min} = e^{-1}$. The highest complexity corresponds in this case to $\bar{\gamma} = 1$.

Remark that for the same width than a Gaussian distribution ($\sigma = \gamma$), the exponential distribution presents a higher complexity ($C_e/C_g \sim 1.4$).

1.5.3 Complexity in a two-level laser model

One step further, combining the results obtained in the former sections, is now done. We calculate LMC complexity for an unrealistic and simplified model of laser [46].

Let us suppose a laser of two levels of energy: $E_1 = 0$ and $E_2 = \varepsilon$, with N_1 atoms in the first level and N_2 atoms in the second level, and the condition $N_1 + N_2 = N$ (the total number of atoms). Our aim is to sketch the statistics of this model and to introduce the results of photon counting [47] that produces an asymmetric behavior of C as function of the population inversion $\eta = N_2/N$. In the range $\eta \in (0, 1/2)$ spontaneous and stimulated emission can take place, but only in the range $\eta \in (1/2, 1)$ the condition to have lasing action is reached, because the population must be, at least, inverted, $\eta > 1/2$.

The entropy S of this system vanishes when N_1 or N_2 is zero. Moreover, S must be homegenous of first order in the extensive variable N [48]. For the sake of simplicity we approach S by the first term in the Taylor expansion:

$$S \sim \kappa\frac{N_1 N_2}{N} = \kappa N\eta(1 - \eta). \tag{1.45}$$

The internal energy is $U = N_2\varepsilon = \varepsilon N\eta$ and the statistical temperature is:

$$T = \left(\frac{\partial S}{\partial U}\right)_N^{-1} = \frac{\varepsilon}{\kappa}\frac{1}{(1 - 2\eta)}. \tag{1.46}$$

Note that for $\eta > 1/2$ the temperature is negative as corresponds to the stimulated emission regime dominating the actual laser action.

We are now interested in introducing qualitatively the results of laser photon counting in the calculation of LMC complexity. It was reported in [47] that the photo-electron distribution of laser field appears to be poissonian. In the continuous limit the Poisson distribution is approached by the normal distribution [49]. The width σ of this energy distribution in the canonical ensemble is proportional to the statistical temperature of the system. Thus, for a switched on laser in the regime $\eta \in [1/2, 1]$, the width of the gaussian energy distribution can be fitted by choosing $\sigma \sim -T \sim 1/(2\eta - 1)$ (recall that $T < 0$ in this case). The range of variation of σ is $[\sigma_\infty, \sigma_{min}] = [\infty, (2\pi e)^{-1/2}]$. Then we obtain:

$$\sigma \sim \frac{(2\pi e)^{-1/2}}{2\eta - 1}. \tag{1.47}$$

By replacing this expression in Eq. (1.42), and rescaling by a factor proportional to entropy, $S \sim \kappa N$, (in order to give to it the correct order of magnitude), LMC complexity for a population inversion in the range $\eta \in [1/2, 1]$ is reobtained:

$$C_{laser} \simeq \kappa N \cdot (1 - 2\eta) \log(2\eta - 1). \tag{1.48}$$

We can consider at this level of discussion $C_{laser} = 0$ for $\eta < 1/2$. Regarding the behavior of this function, it is worth noticing the value $\eta_2 \simeq 0.68$ where the laser presents the highest complexity. By following theses ideas, if the width, σ, of the experimental photo-electron distribution of laser field is measured, the population inversion parameter, η, would be given by Eq. (1.47). In a next step, the LMC complexity of the laser system would be obtained by Eq. (1.48).

It is necessary to remark that a model helps us to approach the reality and provides invaluable guidance in the goal of a finer understanding of a physical phenomenon. From this point of view the present calculation evidently only tries to enlighten the problem of calculating the LMC complexity of a physical system via an unrealistic but simplified model.

1.6 Conclusions

A definition of complexity (LMC complexity) based on a probabilistic description of physical systems has been explained. This definition contains basically an interplay between the *information* contained in the system and the *distance to equipartition* of the probability distribution representing the system. Besides giving the main features of an intuitive notion of complexity, we show that it allows to successfully discern situations considered as complex in systems of a very general interest. Also, its relationship with the Shannon information and the generalized Rényi entropies has been shown to be explicit. Moreover it has been possible to establish the de-

crease of this magnitude when a general system evolves from a near-equilibrium situation to the equipartition.

From a practical point of view, we are convinced that this statistical complexity measure provides a useful way of thinking [50] and it can help in the future to gain more insight on the physical grounds of models with potential biological interest.

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